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Stability and Tunability of an Adaptive Controller for One-Dimensional Parabolic PDE with Spatially Varying Coefficients

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This paper presents tunability analysis for a proposed model reference adaptive control algorithm for linear, one-dimensional, parabolic partial differential equations. Unknown parameters in the known system structure are either constant or spatially-varying, and distributed actuation and sensing are assumed to be available. The adaptation laws are obtained by the Lyapunov redesign method. It is shown that the concept of persistency of excitation in infinite dimensional adaptive systems should be investigated in relation to time variable, spatial variable, and boundary conditions as well. It is shown that even a constant input signal can be sufficiently rich in infinite dimensional adaptive systems in the sense that it can guarantee the convergence of parameter errors to zero.

Key Words: Adaptive Control, Distributed Parameter Systems, Parabolic Systems, Stability, Tunability.

1. Introduction

With the advance of computing technology and related engineering such as composite material and distributed sensing/actuation, many researchers have recently focused on controlling distributed parameter systems(DPS's) (Balas, 1982; Balakrishnan, 1991; Bentsman and Hong, 1991; Bentsman et al., 1991; Helmicki and coauthors, 1992) to name a few.

Compared to the finite dimensional case, the adaptive control of infinite dimensional systems is not well understood and has only recently been studied. One of the main difficulties in synthesizing adaptive control algorithms for DPS's is in guaranteeing the stability of adaptive system. And several new stability criteria in relation to adaptive control have been appeared in the literature (Hong and Bentsman, 1992a; Hong and Wu, 1992; Wu and Hong 1992). Algorithms for self-tuning regulator for DPS's with a finite unknown parameter set have been proposed in Hamza and Sheirah(1978) and Vajta and Keviczky(1981), but no stability proofs have been given. In Balas(1983) some of the possible directions of investigation and the main areas of difficulty for infinite dimensional adaptive control were surveyed. Wen (1985) proposed adaptive control laws and analyzed the Lagrange stability of direct model reference adaptive control(MRAC) schemes in infinite dimensional Hilbert space by using the command generator tracker approach. Finite dimensional adaptive controllers for DPS's were proposed and stabiltity analysis was carried out in Kobayashi(1988) for spectral systems and in Miyasato(1990) for parabolic systems. Some researchers have particularly emphasized direct adaptive control of parabolic partial differential equations (Miyasato, 1990; Hong and Bentsman, 1992b, c; Bentsman et al., 1992).

Parabolic partial differential equations(PDE's) arise in many physical, biological, and engineering problems. For instance the areas of heat transfer, nuclear reactor dynamics, chemical reactions, crystal growth, population genetics, flow of

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electrons and holes in a semiconductor, nerve axon equations, hydrology, petroleum recovery area, and fluid mechanics (Navier-Stokes equations) all are described by parabolic PDE's. To be more specific let us consider the following simple one-dimensional parabolic PDE.

$$\sigma(x)\frac{\partial u}{\partial t}(x, t) = \frac{\partial}{\partial x}\left(\kappa(x)\frac{\partial u}{\partial x}(x, t)\right) + q(x, t, u). \quad (1)$$

In the heat transfer $u\ (x, t)$ is the temperature at spatial coordinate x and at time t, $\sigma(x)$ is the specific heat per unit volume, and $\kappa(x)$ is the characteristic of the material of the rod which is called the heat conductivity. $q(x, t, u)$ is the rate of production of heat energy per unit volume per unit time. q could be either a source or a sink depending on the direction of heat flux. The Eq. (1) is nonlinear in general, however there are interesting cases in which it is of the linear form. (i) q is independent of the local temperature, i. e. $q(x, t, u) = q(x, t)$. An example is the heating caused by an electric current flowing in the rod. (ii) q is proportional to the local temperature, i. e. $q(x, t, u) = q(x, t)u(x, t)$. An example is the heating due to a chemical reaction that takes place at a rate proposional to the local temperature. (iii) Combination of the above two cases, i. e. $q(x, t, u) = b(x, t)u + q(x, t)$.

Another example of (1) in hydrology is fluid flow through porous media, such as soils and groundwater, and oil reservoirs. It is almost impossible in this case to know exact knowledge of the physical parameters of the system by experimental method (Giudici, 1989). Specifically when (1) describes water flow through a saturated soil, u is the piezometric head, $\sigma(x)$ is the storativity for a confined aquifer and porosity for an unconfined one, $\kappa(x)$ is the transmissivity, and q represents the water flux drawn by pumping wells or for a phreatic aquifer. On the other hand if (1) describes water flow through unsaturated soils, u is the hydraulic head, $\sigma(x)$ is the moisture capacity function, $\kappa(x)$ is the hydraulic conductivity, and q is the water uptake by plant root system.

As the above examples, systems of the form (1) appear in many physical and engineering problems. Therefore it is quite worth to investigate issues involved in controlling uncertain systems of parabolic type which has the structure of the form (1).

In this paper we derive and analyze an algorithm for adaptive control of a class of distributed parameter systems described by linear, one-dimensional, parabolic PDE's with unknown coefficients. As in adaptive control of finite dimensional systems, we will focus on the MRAC under the assumption that the structure of the plant is known and only plant parameters of fixed type (time-invariant), not in the boundary condition, are unknown.

Adaptive control problem generally involves two questions : (i) stability of the closed loop system together with proposed adaptation laws, which is often answered by considering an appropriate Lyapunov functional, (ii) convergence analysis of adaptable parameters in the controller to their nominal values. Parameter convergence is closely related to the property of plant input.

The paper has the following structure. In Section 2, we present the stability of an adaptive system. We show the convergence of state error to zero, and the boundedness of all the signals in the closed loop. In Section 3, adjustable parameters in the adaptive controller are shown to converge to their nominal values when an appropriate reference signal is used. Computer simulations are given in Section 4. Conclusions are given in Section 5.

$\Omega = (0, 1)$. $R^+ = [0, \infty)$. $C^k(\Omega)$ is the space of k-times continuously differentiable functions on $\bar{\Omega}$, $k \geq 0$, integer. $L^2(\Omega)$=space of functions square integrable on Ω, where the inner product is denoted by $\langle \bullet, \bullet \rangle$, and $\| \bullet \|$ is the corresponding norm. $\mu(E)$ is the Lebesgue measure of a set $E \subseteq \bar{\Omega}$.

2. Stability of an Adaptive Law

Consider a class of distributed parameter systems described by a linear parabolic partial differential equation with spatially- varying coefficients

$$\frac{\partial u(x,t)}{\partial t}=\frac{\partial}{\partial x}\left(a(x)\frac{\partial u(x,t)}{\partial x}\right)+b(x)u(x,t)+f(x,t),\quad x\in(0,1),\ t>0, \tag{2}$$

where $f(x,t)$ is a control input function. Boundary conditions are given as

$$u(0,t)=\beta_1(t),\ u(1,t)=\beta_2(t). \tag{3}$$

Initial condition is given as

$$u(x,0)=u_0(x). \tag{4}$$

The output y of the measurement system is written in general by

$$y(x_p,t)=G\ u(x,t),\quad x_p\in\Omega_p,\ \Omega_p\subseteq\bar{\Omega},\ t\geq 0, \tag{5}$$

where $G: C([0,1]\times R^+)\rightarrow C(\Omega_p\times R^+)$ is a linear bounded time invariant operator with the form depending on the characteristics of the sensor. Ω_p denotes a subset of $\bar{\Omega}$ where y is defined. In this paper we assume that the system state $u(x,t)$ can be measured at all points of $x\in\bar{\Omega}$ and $t\geq 0$. The following assumptions are made.

Assumptions : (i) The structure of (2) (plant) is a priori known. (ii) Data in the boundary conditions are a priori known. $\beta_1(\bullet)$, $\beta_2(\bullet)$, and $u_0(\bullet)$ are analytic in appropriate domains. (iii) The observation operator G is a priori known ; we may assume that $G=I$, where I denotes the identity operator from $C(\bar{\Omega}\times R^+)$ onto itself. (iv) Coefficients $a(x)$, $b(x)$ are unknown, however it is assumed that $a(x)>0$, and that $a(\bullet)$, $b(\bullet)$ are analytic in $\bar{\Omega}$.

Now the reference model can be taken as

$$\begin{aligned}\frac{\partial u_m(x,t)}{\partial t}&=\frac{\partial}{\partial x}\left(a_m(x)\frac{\partial u_m(x,t)}{\partial x}\right)+b_m(x)u_m(x,t)+r(x,t),\\ u_m(x,0)&=u_{m0},\ t>0,\\ &=A_m u_m(x,t)+r(x,t),\end{aligned}$$

where $$A_m\triangleq\frac{\partial}{\partial x}\left(a_m(x)\frac{\partial}{\partial x}\right)+b_m(x),$$

$$y_m(x_p,t)=G\ u_m(x,t),\ x_p\in\Omega,\ t\geq 0. \tag{6}$$

where $r(x,t)$ is the reference input. The subscript m indicates variables and parameters related to the reference model. Noting that the reference model is at our disposal, we assume that $a_m(x)\geq a_0>0$, $b_m(x)<0$, $|b_m(x)|\geq b_0>0$, and that $a_m(x)$, $b_m(x)$ are analytic in Ω. The boundary conditions of the reference model are assumed to be the same as (3). It is known that any solution of (6) with (3) is analytic in $\bar{\Omega}\times\{0<t<T<\infty\}$, if $r(\bullet,\bullet)$, $a_m(\bullet)$, and $b_m(\bullet)$ are analytic in their appropriate domains (Friedman, 1969, p. 212).

The control objective for MRAC can now be stated as follows : find a bounded control signal $f(x,t)$ that drives $u(x,t)$ to $u_m(x,t)$ asymptotically and keeps all signals in the closed loop bounded.

Now let us consider the following control law $f(x,t)$ with adjustable(adaptive) parameters $\phi_a(x,t)$ and $\phi_b(x,t)$ such that

$$f(x,t)=\frac{\partial}{\partial x}\left(\phi_a(x,t)\frac{\partial u(x,t)}{\partial x}\right)+\phi_b(x,t)u(x,t)+r(x,t). \tag{7}$$

The adaptation law is given as

$$\frac{\partial\phi_a(x,t)}{\partial t}=\varepsilon\frac{\partial e(x,t)}{\partial x}\frac{\partial u(x,t)}{\partial x},\quad \phi_a(x,0)=\phi_{a0}, \tag{8}$$

$$\frac{\partial\phi_b(x,t)}{\partial t}=-\varepsilon e(x,t)u(x,t),\quad \phi_b(x,0)=\phi_{b0}, \tag{9}$$

where $\varepsilon>0$ is called the adaptation gain.

Theorem 1 : Consider a parabolic plant (2)-(4) with the assumptions above, and let the reference model be given by (6). Let the feedback control law $f(x,t)$ be given as (7) with the adaptation law (8)-(9). Then all the signals in the colsed loop system are bounded and $\|e(x,t)\|\rightarrow 0$ as $t\rightarrow\infty$.

We first introduce the following Lemmas to be used subsequently in proving Theorem 1.

Lemma 1 : (Popov, 1973, p. 211) : If $f(t): R^+\rightarrow R$ is uniformly continuous for $t\geq 0$, and $\lim_{t\to\infty}\int_0^t|f(\tau)|d\tau$ exists and is finite, then $\lim_{t\to\infty}f(t)=0$.

The following is an extension of Lemma 1 to functions with two independent variables.

Lemma 2 : If $e(x,t):[0,1]\times R^+\rightarrow R$ is bounded, $\{e(x,t)\}_{x\in[0,1]}$ is equicontinuous in t, and $\lim_{t\to\infty}\int_0^t\|e(x,\tau)\|^2 d\tau$ exists, and is finite, then $\lim_{t\to\infty}\|e(x,t)\|=0$.

Proof : Let the bound for $e(x,t)$ be M. From the equicontinuity, for any $\varepsilon>0$ there exists $\delta(\varepsilon)>0$

such that whenever $|t_1-t_2|<\delta$ for every $x\in[0, 1]$ we have

$$|e(x, t_1)-e(x, t_2)|<\frac{\varepsilon}{2M}. \tag{10}$$

Now,

$$|e^2(x, t_1)-e^2(x, t_2)|=|e(x, t_1)+e(x, t_2)|$$
$$|e(x, t_1)-e(x, t_2)|\le 2M\cdot\frac{\varepsilon}{2M}=\varepsilon.$$

Therefore $\{e^2(x, t)\}_{x\in[0, 1]}$ is equicontinuous in t. Define

$$f(t)\triangleq\|e(x, t)\|^2. \tag{11}$$

Then

$$|f(t_1)-f(t_2)|=\|e(x, t_1)\|^2-\|e(x, t_2)\|^2$$
$$|\le\int_0^1|e^2(x, t_1)-e^2(x, t_2)|dx\le\varepsilon.$$

Hence $f(t)$ is uniformly continuous. Since $f(t)$ satisfies both hypotheses in Lemma 1, $\|e(x, t)\|\to 0$ as $t\to\infty$. Q.E.D.

Corollary 1: If $e(x, t)\in L^2([0, 1]\times R^+)\cap L^\infty([0, 1]\times R^+)$, and $\frac{\partial e(x, t)}{\partial t}$ is bounded, then $\|e(x, t)\|\to 0$ as $t\to\infty$.

Proof: The proof follows directly from Lemma 2.

Lemma 3 (Pazy, 1983, p. 114): Consider an abstract Cauchy problem on a Banach space X as

$$\frac{\partial u(t)}{\partial t}=Au(t)+f(t), \ u(0)=u_0. \tag{12}$$

Let A be the infinitesimal generator of an analytic semigroup $T(t)$, and let $f\in C^\alpha([0, T]: X)$, $\alpha\in(0, 1)$, $T>0$, and let u be the solution of the initial value problem (12) on $[0, T]$. If $u_0\in D(A)$, then Au, $\frac{du}{dt}$ are continuous on $[0, T]$.

Proof of Theorem 1: Let us define the state error e and controller parameter errors ψ_a and ψ_b as

$$e(x, t)\triangleq u(x, t)-u_m(x, t),$$
$$e(0, t)=e(1, t)=0, \tag{13}$$
$$\psi_a(x, t)\triangleq\phi_a(x, t)-\phi_a^*(x), \tag{14}$$
$$\psi_b(x, t)\triangleq\phi_b(x, t)-\phi_b^*(x), \tag{15}$$

where ϕ_a^* and ϕ_b^* are unknown functions which are defined as

$$\phi_a^*(x)=a_m(x)-a(x), \tag{16}$$
$$\phi_b^*(x)=b_m(x)-b(x). \tag{17}$$

Note that $\partial\psi_a(x, t)/\partial t=\partial\phi_a(x, t)/\partial t$, and $\partial\psi_b(x, t)/\partial t=\partial\phi_b(x, t)/\partial t$. The point is that when controller parameters $\phi_a(x, t)$, $\phi_b(x, t)$ in (8)-(9) converge to their nominal values ϕ_a^* and ϕ_b^*, respectively, the closed loop state of the plant matches the state of the reference model exactly, i. e.

$$\begin{aligned}\frac{\partial u(x, t)}{\partial t}&=\frac{\partial}{\partial x}\left(a(x)\frac{\partial u(x, t)}{\partial x}\right)+b(x)u(x, t)\\&+\left[\frac{\partial}{\partial x}\left(\phi_a(x, t)\frac{\partial u(x, t)}{\partial x}\right)\right.\\&\left.+\phi_b(x, t)u(x, t)+R(x, t)\right]\\&=\frac{\partial}{\partial x}\left((a(x)+\phi_a(x, t)\frac{\partial u(x, t)}{\partial x}\right)+(b(x)\\&+\phi_b(x, t))u(x, t)+r(x, t),\end{aligned} \tag{18}$$

becomes the reference model when $\lim_{t\to\infty}\phi_a(x, t)=\phi_a^*$, and $\lim_{t\to\infty}\phi_b(x, t)=\phi_b^*$. Subtracting (6) from (18) we have the state error equation as

$$\begin{aligned}\frac{\partial e(x, t)}{\partial t}&=\frac{\partial u(x, t)}{\partial t}-\frac{\partial u_m(x, t)}{\partial t}\\&=\frac{\partial}{\partial x}\left(a_m(x)\frac{\partial e(x, t)}{\partial x}\right)+b_m(x)e(x, t)\\&+\left[\frac{\partial}{\partial x}\left(\psi_a(x, t)\frac{\partial u(x, t)}{\partial x}\right)+\psi_b(x, t)u(x, t)\right].\end{aligned} \tag{19}$$

Consider a Lyapunov functional as

$$V(e, \psi_a, \psi_b)=\frac{1}{2}\int_0^1\left[e^2(x, t)+\frac{1}{\varepsilon}(\psi_a^2(x, t)+\psi_b^2(x, t))\right]dx. \tag{20}$$

Differentiating V with respect to t along the trajectories of (19) employing integration by parts applying boundary conditions, and utilizing the adaptation law (8)-(9) yields

$$\begin{aligned}\frac{\partial V}{\partial t}&=\int_0^1\left[e\left\{\frac{\partial}{\partial x}\left(a_m\frac{\partial e}{\partial x}\right)+b_m e+\frac{\partial}{\partial x}\left(\psi_a\frac{\partial u}{\partial x}\right)+\psi_b u\right\}\right.\\&\left.+\frac{1}{\varepsilon}\left(\psi_a\frac{\partial\psi_a}{\partial t}+\psi_b\frac{\partial\psi_b}{\partial t}\right)\right]dx\\&=\int_0^1\left[-a_m(x)\left(\frac{\partial e(x, t)}{\partial x}\right)^2+b_m(x)e^2(x, t)\right]dx\\&\le-\left(\frac{a_0\pi^2}{16}+b_0\right)\int_0^1e^2(x, t)dx, \qquad (21)\\&\le 0. \qquad (22)\end{aligned}$$

The inequality in (21) is achieved by the fact that for a linear set of functions continuous with their first derivatives in the closed interval [a, b], the Friedrichs inequality (Rektorys, 1980) is given as

$$\int_a^b h^2(x)dx \le c_1\int_a^b\left(\frac{\partial h(x)}{\partial x}\right)^2 dx + c_2 h^2(a),$$

where $c_1=16(b-a)^2/\pi^2$, and $c_2=4(b-a)/\pi$. Since $V(e, \psi_a, \psi_b)$ is non-increasing and bounded below, $\psi_a(x, t)$, and $\psi_b(x, t)$ are bounded in the norm $\|\bullet\|$. Also from $\dot{V}\le -C\|e(x, t)\|^2$, $C>0$, we have

$$0\le\int_{t_0}^{\infty}\|e(x, \tau)\|^2 d\tau \le \frac{V(t_0)-V(\infty)}{C}<\infty. \quad (23)$$

Consequently, it is shown that $\|e(x, t)\|\in L^2[0, \infty)$. Since u_m is bounded for bounded input r, so is u from (13) for all $t>0$. ϕ_a and ϕ_b are all bounded from (14) (15) due to the boundedness of ψ_a and ψ_b. Since the initial states of ϕ_a and ϕ_b in (8) (9) can be chosen appropriately, the solution u of (18) is analytic for all $t>0$. Finally the boundedness of $\partial e(x, t)/\partial t$ in (19) for all $t>0$ is assured from Lemma 3 together with the fact that A_m in (6) generates and exponentially stable semigroup, and the analyticity of u, ψ_a and ψ_b. Therefore the conclusion of the theorem follows from Corollary 1. Q.E.D.

Remark : In a special case when the coefficients $a(x)$ and $b(x)$ in (2) are constants rather than spatially-varying, the Lyapunov functional can be taken as

$$V=\frac{1}{2}<e, e>+\frac{1}{2\varepsilon}(\psi_a^2(t)+\psi_b^2(t)), \quad (24)$$

and we can proceed similarly to the case of spatially-varying coefficients. Then the control law becomes

$$f(x, t)=\phi_a(t)\frac{\partial^2 u(x, t)}{\partial x^2}+\phi_b(t)u(x, t) + r(x, t), \quad (25)$$

with the adaptation laws as

$$\frac{\partial\psi_a(t)}{\partial t}=\varepsilon<\frac{\partial e(x, t)}{\partial x}, \frac{\partial u(x, t)}{\partial x}>,$$
$$\frac{\partial\psi_b(t)}{\partial t}=-\varepsilon<e(x, t), u(x, t)>. \quad (26)$$

Remark : Equations (19) and (8)-(9) represent the overall adaptive system. By substituting (8)-(9) into (19) the state error system (19) has the form

$$\frac{\partial e(x, t)}{\partial t}=\frac{\partial}{\partial x}\left(\alpha(x, t, \frac{\partial e}{\partial x})\frac{\partial e(x, t)}{\partial x}\right)+g(x, t, e, \frac{\partial e}{\partial x}), \quad (27)$$

where

$$\alpha(x, t, \frac{\partial e}{\partial x})=a_m(x)+\psi_a(x, 0) + \varepsilon\int_0^t\left(\frac{\partial e}{\partial x}\frac{\partial e}{\partial x}+\frac{\partial e}{\partial x}\frac{\partial u_m}{\partial x}\right)dt, \quad (28)$$

$$g(x, t, e, \frac{\partial e}{\partial x})=\frac{\partial}{\partial x}\left(\{\alpha(x, t, \frac{\partial e}{\partial x})-a_m(x)\}\frac{\partial u_m}{\partial x}\right) + \{\psi_b(x, 0)-\varepsilon\int_0^t e(e+u_m)dt\}(e+u_m)+b_m e. \quad (29)$$

The initial conditions of (8) needs to be chosen such that $a_m(x)+\psi_a(x, 0)>0$. Since the exogeneous signal u_m is smooth, there exists a $t_0>0$ such that the principal part of (27) $\frac{\partial}{\partial x}\left(\alpha(x, t, \frac{\partial e}{\partial x})\frac{\partial}{\partial x}\right)$ is strongly elliptic for all $t\in[0, t_0]$. Therefore (27) is parabolic (Friedman, 1969, p. 179 and p. 134). Hence the results of (Friedman, 1969, pp. 169~181) are applicable for the existence of a unique solution of (27) for $t\in[0, t_0]$. Specifically the A_0 on the page 169 of (Friedman, 1969) is $\frac{\partial}{\partial x}\left(\{a_m(x)+\psi_a(x, 0)\}\frac{\partial}{\partial x}\right)$, and satisfaction of the conditions $F2-F4$ of (Friedman, p. 169~170) is easily seen by choosing those α, σ, ρ on the p. 170 of (Friedman, 1969) as $\alpha=1/2$, and $\sigma=\rho=1$ in our case. Finally the Lyapunov function defined as in (20) ensures that all soultions belong to a closed bounded set, their existence for all $t\ge 0$ is guaranteed as well.

3. Tunability Analysis

The purpose of this section is to illustrate one of main differences between the adaptive control of distributed parameter systems and that of finite dimensional systems.

In the case of MRAC of lumped parameter systems when no modeling errors are present in the model of the plant, the knowledge of the upper bound of the order of the plant enables us to develop the controller structure and adaptive laws, and to prove that the output of the plant tracks the output of the model asymptotically and that all signals in the adaptive loop are uniformly bounded (Narendra and Annaswamy, 1989 ; Sastry and Bodson, 1989), where strictly positive realness(or signal dependent positivity condition) of the reference model is assumed to guarantee the stability. Furthermore, when the order of the plant is exactly known and no pole-zero cancellation exists, a necessary and sufficient condition for the exponential convergence of parameters and the tracking error to zero is that the measured vector signal (regressor vector) is persistently exciting (PE). The PE property of the filtered signals is guaranteed by selecting the reference input signal to have a certain number of frequencies (Sastry and Bodson, p. 90). The PE property is used to establish robustness of adaptive systems in the presence of bounded disturbances and unmodeled dynamics. Furthermore, Rohrs et al.(1985) discovered that the bounded-input bounded-state stability is not robust with respect to uncertainties, and showed examples that an arbitrarily small disturbance can destabilize an adaptive system. The exponential stability of adaptive system is important in the sense that an exponentially stable system can tolerate a certain amount of disturbances and unmodeled dynamics. In the finite dimensional adaptive control the exponential stability of the nominal adaptive system is guaranteed by the PE condition of reference input.

In distributed parameter case, we are still far from complete understanding of the nature of an adaptive system, but if we relate the concept of PE to the convergence of the parameters in the controller to their nominal values, where obtaining nominal values guarantees the exact model matching, the PE property should be investigated through time variable t, spatial variable x, and furthermore boundary conditions. In the finite dimensional case, for example, when the reference input is just constant, it is not PE, therefore the parameter error does not converge to zero when constant input is used, but in the infinite dimensional case even constant input could be PE. An example in (Hong and Bentsman, 1992c) shows that if at least one of the Fourier coefficient of the reference input is not zero, then the controller parameters are shown to converge to their nominal values in the case of constant coefficient parabolic systems. Without loss of generality, it will be assumed in this section that the initial function $u_0(x)$ is known since distributed measurement is possible.

Definition 1 : Let $\{\phi_i\}$ be the set of adjustable parameters in the controller such that $\lim_{t\to\infty}\phi_i=\phi_i^*$ for every i imply $\lim_{t\to\infty} e=0$. Then ϕ_i is said to be tunable if $\lim_{t\to\infty}\phi_i=\phi_i^*$ for $\lim_{t\to\infty} e=0$.

We adopt the approach by Kitamura and Nakagiri(1977), and begin the analysis with the assumption that $e(x, t)=0$ was achieved.

Lemma 4 : The state error $e(x, t)$ is equal to zero for all $x\in[0, 1]$ and all $t\geq 0$ if and only if

$$\frac{\partial}{\partial x}\left(\psi_a(x, t)\frac{\partial u(x, t)}{\partial x}\right)+\psi_b(x, t)u(x, t)=0, \tag{30}$$

for all $x\in(0, 1)$ and all $t>0$, where ψ_a and ψ_b are defined in (2.12a, b).

Proof : From (19) the necessity part directly follows. If (30) holds, (19) becomes

$$\begin{aligned}&\frac{\partial e(x, t)}{\partial t}=\frac{\partial}{\partial x}\left(a_m(x)\frac{\partial e(x, t)}{\partial x}\right)+b_m(x)e(x, t),\\&e(x, 0)=0,\\&e(0, t)=e(1, t)=0,\end{aligned}\tag{31}$$

for all $x\in(0, 1)$. Due to the uniqueness of the solution, the sufficiency follows. Q.E.D.

Lemma 5 : Assume that $\lim_{t\to\infty} e(x, t)=0$. If the functions $\frac{\partial^2 u}{\partial x^2}(x, t)$, $\frac{\partial u}{\partial x}(x, t)$, and $u(x, t)$ are linearly independent as functions of t on a dense subset in $\bar{\Omega}$, then $\phi_a(x, t)$ and $\phi_b(x, t)$ are tunable.

Proof : From (2.7) with $\lim_{t\to\infty} e(x, t)=0$, we have $\lim_{t\to\infty}\frac{\partial\psi_a(x, t)}{\partial t}=\lim_{t\to\infty}\frac{\partial\psi_b(x, t)}{\partial t}=0$, hence $\lim_{t\to\infty}\psi_a(x, t)$ and $\lim_{t\to\infty}\psi_b(x, t)$ exist. Therefore from (30) as

$$\begin{aligned}&\psi_a(x, \infty)\frac{\partial^2 u}{\partial x^2}(x, \infty)+\frac{\partial\psi_a}{\partial x}(x, \infty)\frac{\partial u}{\partial x}(x, \infty)\\&+\psi_b(x, \infty)u(x, \infty)=0,\end{aligned}\tag{32}$$

and linear independency, we have $\psi_a(x, \infty)=\psi_b(x, \infty)=0$ on some dense set in $\bar{\Omega}$. By continuity, $\psi_a(x, \infty)=\psi_b(x, \infty)=0$ for all $x\in\bar{\Omega}$. Q.E.D.

To be more specific we give the following example. Since for spatially-varying coefficients it is cumbersome to obtain an explicit form of the solution, we consider constant coefficients case.

Theorem 2 : Assume that $a(x)=a$, $b(x)=b$, a, b constants, and $e(x, t)=0$. Let the reference input is given by $r(x, t)=r(x)$. Then $\phi_a(t)$, $\phi_b(t)$ are tunable if either

(i) $r(x)\neq 0$, or
(ii) $\beta_1(t)\neq 0$, or
(iii) $\beta_2(t)\neq 0$.

Proof : Since $e(x, t)=0$, $u(x, t)$ can be replaced by the state of the reference model $u_m(x, t)$. The solution $u_m(x, t)$ of (6) with (3) when $a(x)=a$, $b(x)=b$ is given as

$$\begin{aligned} u_m(x, t)=&\sum_{n=1}^{\infty}<u_{mo}, \psi_n>e^{-k_nt}\psi_n(x)\\ &+\int_0^t\int_0^1\Big(\sum_{n=1}^{\infty}e^{-k_n(t-\tau)}\psi_n(x)\psi_n(y)\Big)r(y, \tau)dyd\tau\\ &+\int_0^t\Big(\sum_{n=1}^{\infty}(\psi_n(1)-\psi_n'(1))e^{-k_n(t-\tau)}\psi_n(x)\Big)\beta_2(\tau)d\tau\\ &+\int_0^t\Big(\sum_{n=1}^{\infty}(\psi_n(0)-\psi_n'(0))e^{-k_n(t-\tau)}\psi_n(x)\Big)\beta_1(\tau)d\tau, \end{aligned} \tag{33}$$

where k_n, $\psi_n(x)$ are eigenvalues and corresponding eigenfunctions, and the prime stands for the derivative with respect to x. By setting $\beta_1(t)=\beta_2(t)=0$, $u_{m0}=0$ and $r(x, t)=r(x)$, (33) yields

$$\begin{aligned} u_m(x, t)&=\sum_{n=1}^{\infty}e^{-k_nt}\psi_n(x)\int_0^t e^{k_n\tau}\Big(\int_0^1\psi_n(y)r(y)dy\Big)d\tau.\\ &=\sum_{n=1}^{\infty}\frac{1}{k_n}(1-e^{-k_nt})<\psi_n, r>\psi_n(x), \end{aligned} \tag{34}$$

where $<\psi_n, r>\triangleq\int_0^1\psi_n(y)r(y)\,dy$, and $\psi_n(x)=\sin\lambda_nx$. Therefore,

$$\begin{aligned} u_m(x, t)&=\sum_{n=1}^{\infty}\frac{1}{k_n}(1-e^{-k_nt})<\psi_n, r>\sin\lambda_nx=\sum_{n=1}^{\infty}Q_n(t)\sin\lambda_nx\,;\\ \frac{\partial u_m(x, t)}{\partial x}&=\sum_{n=1}^{\infty}\lambda_nQ_n(t)\cos\lambda_nx\,;\\ \frac{\partial^2u_m(x, t)}{\partial x^2}&=-\sum_{n=1}^{\infty}(\lambda_n)^2Q_n(t)\sin\lambda_nx. \end{aligned}$$

For α, β, $\gamma\in R$,

$$\begin{aligned} 0&=\alpha u_m+\beta\frac{\partial u_m(x, t)}{\partial x}+\gamma\frac{\partial^2u_m(x, t)}{\partial x^2},\\ &=\sum_{n=1}^{\infty}\frac{1}{k_n}(1-e^{-k_nt})<\psi_n, r>\Big[(\alpha-\gamma\lambda_n^2)\sin\lambda_nx+\beta\lambda_n\cos\lambda_nx\Big], \end{aligned} \tag{35}$$

implies $\alpha=\beta=\gamma=0$. Therefore u_m, $\frac{\partial u_m}{\partial x}$, $\frac{\partial^2u_m}{\partial x^2}$ are linearly independent in t, for all $x\in[0, 1]$. Conclusion ii) and iii) in the theorem can be drawn similarly from the third and fourth term, respectively, in (33). Therefore from Lemma 3. 2, the proof follows. Q.E.D.

Remark : Theorem 2 suggests that in infinite dimensional adaptive system, the PE property of input signals should be investigated through temporal variable, spatial variable, and boundary conditions. Conclusions of Theorem 2 may hold for the case of spatially-varying coefficients as well.

Definition 2 : Let $g(x)$ be appropriately smooth. Define $E_i=\{x\in\Omega:\frac{d^ig(x)}{dx^i}=0\}$. Then $g(x)$ is said to be i-th order persistently exciting if $E_i\neq\emptyset$, and $\mu(E_i)=0$.

Theorem 3 : Let $b(x)=0$. Then $\psi_a(x, t)$ is tunable if $u(x, t)$ is of first order persistently exciting for all $t>0$.

Proof : From (30) with $b(x)=0$ we have

$$\frac{\partial}{\partial x}\Big(\psi_a(x, t)\frac{\partial u(x, t)}{\partial x}\Big)=0,$$

for all $x\in\Omega$ and all $t>0$.

By integrating

$$\psi_a(x, t)\frac{\partial u(x, t)}{\partial x}=f(t)$$

where $f(t)$ is only a function of t. Since $E_1\neq\emptyset$ for each $t>0$, $f(t)=0$. On the other hand since $\mu(E_1)=0$, the set $\{x\in\Omega:\psi_a(x, t)=0\}$ is dense in Ω. By continuity $\psi_a(x, t)=0$. Q.E.D.

Remark : Again the tunability of ψ_a is determined by the profile of u which is in turn related to r and boundary conditions.

4. Computer Simulations

In this section we demonstrate the stability of the adaptive control law (7)-(9) through computer simulations (Convergence analysis via numerical examples is referred to Hong and Bentsman(1992b, c)). The parabolic plant with spatially varying coefficients and homogeneous boundary conditions ($\beta_1(t)=\beta_2(t)=0$) is given as

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}\right)+f(x,\ t),$$
$$u(x,\ 0)=u_0(x),\ x\in[0,\ 1],\ t>0, \qquad (36)$$

where the unknown coefficient $a(x)$ is assumed to be $0.1+0.2\sin(\pi x)$. The initial condition is given

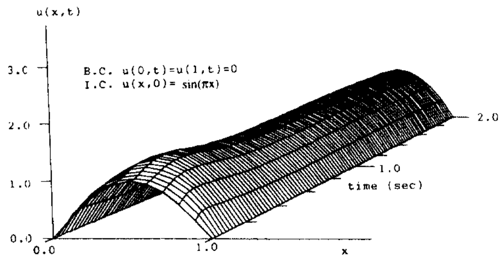

Fig. 1 The output u(x, t) of the plant (36) which follows the output $u_m(x, t)$ of the reference model (37)

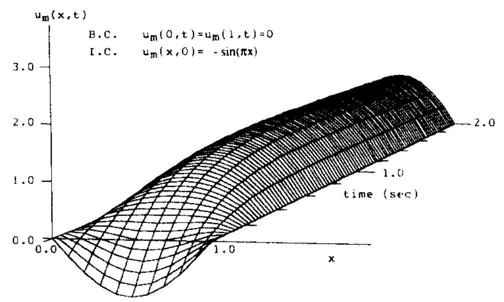

Fig. 2 The output $u_m(x, t)$ of the reference model (37) with reference input $r(x, t)=5.0$

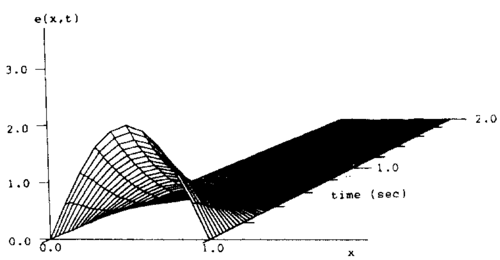

Fig. 3 The state error e(x, t) which converges to zero exponentially

as $u_0(x)=\sin(\pi x)$. The reference model is defined as

$$\frac{\partial u_m}{\partial t}=0.5\frac{\partial^2 u_m}{\partial x^2}+r(x,\ t),u_m(x,\ 0)$$
$$=u_{m0}(x),\ x\in[0,\ 1],\ t>0. \qquad (37)$$

Let $u_{m0}(x)=-\sin(\pi x)$, and the constant reference input be given as $r(x,\ t)=5.0$. Then the adaptive control law (7) and the adaptation law (8) are given as

$$f(x,\ t)=\frac{\partial}{\partial x}\left(\phi_a(x,\ t)\frac{\partial u}{\partial x}\right)+5.0$$
$$\frac{\partial \phi_a(x,\ t)}{\partial t}=\varepsilon\frac{\partial e(x,\ t)}{\partial t}\frac{\partial u(x,\ t)}{\partial t},$$
$$\phi_a(x,\ 0)=0.1$$

where $\varepsilon=0.1$. Now Fig. 1 shows the output of the plant (36) which follows the output from the reference model (37) which is given in Fig. 2. Figure 3 shows the exponential convergence of the state error $e(x,\ t)$ to zero which in part demonstrates the exponential stability of the adaptive system. The nominal value $\phi_a^*(x)$ of the controller parameter $\phi_a(x,\ t)$ is given as

$$\phi_a^*(x)=a_m(x)-a(x)$$
$$=0.5-0.1+0.2\sin(\pi x))$$
$$=0.4-0.2\sin(\pi x).$$

5. Conclusions

In this paper we considered the direct model reference adaptive control of a class of parabolic PDE's with the assumption that distributed measurement and control were possible. We confirmed that the Lyapunov redesign approach can be extended to the infinite dimensional systems as well. In Section 3 it was shown that the convergence of the parameter error to zero is closely related to the property of the plant output signal which is in turn related to the reference input and boundary conditions. It was shown that the concept of persistency of excitation should be investigated through temporal variable, spatial variable and boundary conditions as well in the adaptive control of distributed parameter systems.

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